SOME RESULTS ON SECOND ORDER DIFFERENTIABILITY IN THE EXTENDED SENSE OF FUNCTIONS

Ha Anh Tuan
Faculty of Basic Science, Ho Chi Minh City University of Transport
Received on 30/6/2021, accepted for publication on 28/9/2021

Abstract: In this paper, we introduce the concept of second order partial derivatives in the extended sense for nonconvex functions and prove a formula computing the extended Hessian in terms of the second order partial derivatives in the extended sense. We show that the sum, difference, product, and quotient of functions that are twice differentiable at a point are functions that are twice differentiable at that point in the extended sense. We also show why the counterpart of the second order differentiability in the extended sense on $\mathbb{R}^n$ does not appear in variational analysis.

Keywords: Twice differentiable in the extended sense; second order partial derivatives in the extended sense.

1 Introduction

Classical second order differentiability is an important property from both theoretical and practical viewpoints. However, since many functions from optimization and its applications do not have this property, various generalized second order differentiation notions have been proposed and studied extensively in the literature ([2], [5]).

In 1998, Rockafellar and Wets [5] introduced the notion of second order differentiability of a function in the extended sense by removing the differentiability in some negligible subsets of a neighborhood of the considered point from the definition of classical second order differentiability. Although the extended second order differentiability is weaker than the classical counterpart, it still ensures the function of having a quadratic expansion. A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is twice differentiable at $\bar{x}$ in the extended sense if and only if f is finite and locally lower semicontinuous at $\bar{x}$ and the subgradient mapping $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is differentiable at $\bar{x}$. Other nice property of such functions can be found in [5, Chapter 13].

Our goal is to develop a system of computational rules for extended Hessian. We have achieved results:

The first result, we have established the extended Hessian expression formula through extended second order partial derivatives.

The second result, we have established the extended Hessian expression formula of the sum, difference, product, and quotient functions of two functions which are twice differentiable at the same point in the extended sense.

Together with the differentiability at a point, the differentiability on $\mathbb{R}^n$ is also a remarkable property from both theoretical and practical viewpoints. So it is curious why the counterpart of the extended second order differentiability does not appear in variational analysis. That is another interesting result we obtained.

[1)] Email:hatuanhuyhoang@gmail.com

2 Preliminaries

This section recalls some notions and their properties from variational analysis ([2], [5]), which are used in the sequel. Let Ω be a subset of the Euclidean space $\mathbb{R}^n$ and $\bar{x} \in \Omega$. Recall [2] that the *regular normal cone* to Ω at $\bar{x} \in \Omega$ is the set $\widehat{N}_\Omega(\bar{x})$ given by

$$\widehat{N}_\Omega(\bar{x}) := \left\{ v \in \mathbb{R}^n \;\Big|\; \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \le 0 \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$; the (Mordukhovich) *limiting/basic normal cone* to Ω at $\bar{x} \in \Omega$ is the set $N_\Omega(\bar{x})$ defined by

$$N_\Omega(\bar{x}) := \big\{ v \in \mathbb{R}^n \,|\, \text{there exists}\, x_k \xrightarrow{\Omega} \bar{x},\; v_k \in \widehat{N}_\Omega(x_k) \text{ with } v_k \to v \big\},$$

which was introduced by Mordukhovich [3] in an equivalent form. If $\bar{x} \notin \Omega$, one puts $\widehat{N}_\Omega(\bar{x}) = \emptyset$ by convention. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and let $\bar{x} \in \text{dom} f := \big\{ x \in \mathbb{R}^n |\; f(x) < \infty \big\}$. Recall [2] that the *limiting subdifferential* (also known as the Mordukhovich/basic subdifferential) of f at $\bar{x}$ is given by

$$\partial f(\bar{x}) := \big\{ v \in \mathbb{R}^n \,|\, (v, -1) \in N_{\text{epi} f}(\bar{x}, \bar{y}) \big\},$$

where $\text{epi}\, f := \big\{ (x, r) \in \mathbb{R}^n \times \mathbb{R} |\; r \ge \varphi(x) \big\}$ is the epigraph of f.
One says f is *Lipschitz* on $U \subset \mathbb{R}^n$ if there exists a real number $\kappa > 0$ such that

$$|f(x) - f(u)| \le \kappa \|x - u\| \quad \text{for all } x, u \in D;$$

if, in addition, U is a neighbourhood of $\bar{x}$, then f is said to be *locally Lipschitz* around $\bar{x}$. If f is locally Lipschitz around $\bar{x}$, the *Clarke subddiferential* of f at $\bar{x}$ is defined as the convex hull of $\partial f(\bar{x})$, and will be denoted by $\partial^{Cl} f(\bar{x})$ in the sequel.
Recall [5] that f is *twice differentiable* at $\bar{x}$ (in the classical sense) if it is differentiable on a neighborhood U of $\bar{x}$ and there exists a $n \times n$ matrix H such that

$$\lim_{x \xrightarrow{U} \bar{x}} \frac{\big[\nabla f(x)\big]^T - \big[\nabla f(\bar{x})\big]^T - H(x - \bar{x})}{\|x - \bar{x}\|} = 0,$$

where $\nabla f(\bar{x})$ is written as a row vector. In this case, the matrix H is necessarily unique, called the Hessian (matrix) of f at $\bar{x}$, and is denoted by $\nabla^2 f(\bar{x})$.

Definition 2.1. Let a function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$. We say that

(**i**) f is *twice differentiable at* $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_i, \ldots, \bar{x}_n)$ *in the extended sense* if it is differentiable at $\bar{x}$, and there exists a $n \times n$ matrix A, a neighborhood U of $\bar{x}$ and a subset D of U with $\mu(U \backslash D) = 0$ such that f is Lipschitz on U, differentiable on D, and

$$\lim_{x \xrightarrow{D} \bar{x}} \frac{\big[\nabla f(x)\big]^T - \big[\nabla f(\bar{x})\big]^T - A(x - \bar{x})}{\|x - \bar{x}\|} = 0,$$

where μ denotes the Lebesgue measure on $\mathbb{R}^n$. This matrix A, necessarily unique, is then called the Hesian (matrix) of f at $\bar{x}$ in the extended sense and is likewise denoted by $\nabla^2 f(\bar{x})$.

(**ii**) f is said that has *second order partial derivatives for x_i at $\bar{x}$ in the extended sense* $(i = 1, 2, \dots, n)$ if f has partial derivatives $\frac{\partial f}{\partial x_j}(\bar{x})$, $(j = 1, 2, \dots, n)$ and

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) := \lim_{x_i \xrightarrow{D_i} \bar{x}_i} \frac{\frac{\partial f}{\partial x_j}(\bar{x}_1, \dots, x_i, \dots, \bar{x}_n) - \frac{\partial f}{\partial x_j}(\bar{x}_1, \dots, \bar{x}_i, \dots, \bar{x}_n)}{x_i - \bar{x}_i}$$

exists $(j = 1, 2, \dots, n)$. Where D_i is a subset of $U_i \subset \mathbb{R}$ satisfies $\mu(U_i \setminus D_i) = 0$ for some neighborhood U_i of $\bar{x}_i$.

From definition, it is easy to see that if f is twice differentiable at $\bar{x}$ then it is twice differentiable at $\bar{x}$ in the extended sense, and the Hessian and the extended Hessian coincide.

Example 2.2. The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^4 & \text{if } x \geq 1, \\ \frac{(2n+1)(2n^2+2n+1)}{n^3(n+1)^3} x + \frac{1}{(n+1)^3} - \frac{1}{n^3} & \text{if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right),\ n = 1, 2, ... \\ 0 & \text{if } x = 0, \\ f(-x) & \text{if } x < 0, \end{cases}$$

is twice differentiable at $\bar{x} = 0$ in the extended sense, but it is not twice differentiable at $\bar{x}$ in the classical sense. Indeed, we see that f is differentiable at $\bar{x}$, and

$$\nabla f(x) = \begin{cases} 4x^3 & \text{if } x > 1, \\ \frac{(2n+1)(2n^2+2n+1)}{n^3(n+1)^3} & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right),\ n = 1, 2, ... \\ 0 & \text{if } x = 0, \\ -\nabla f(-x) & \text{if } \ x \in (-\infty, 0) \setminus \{-\frac{1}{n} |\ n \in \mathbb{N}^*\}. \end{cases}$$

Put $U = (-1, 1)$, $D = (-1, 1) \setminus \{\frac{1}{n} |\ n \in \mathbb{Z}^*\}$, and $A = 0$. Then $\mu(U \setminus D) = 0$, f is Lipschitz on U with constant $\kappa = 1$, and differentiable on D, where μ is the Lebesgue measure on $\mathbb{R}$. Furthermore, for each $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ with $n \in \mathbb{N}^*$ we have

$$\begin{aligned} \left| \frac{\left[\nabla f(x)\right]^T - \left[\nabla f(\bar{x})\right]^T - A(x - \bar{x})}{\|x - \bar{x}\|} \right| &= \frac{(2n+1)(2n^2+2n+1)}{n^3(n+1)^3 |x|} \\ &\leq \frac{(2n+1)(2n^2+2n+1)}{n^3(n+1)^2} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies

$$\lim_{x \xrightarrow{D} \bar{x}} \frac{\left[\nabla f(x)\right]^T - \left[\nabla f(\bar{x})\right]^T - A(x - \bar{x})}{\|x - \bar{x}\|} = 0.$$

Therefore, f is twice differentiable at $\bar{x}$ in the extended sense. On the other hand, since f is not differentiable at each point $\frac{1}{n}$ with $n \in \mathbb{Z}^*$. Hence, f is not twice differentiable at $\bar{x}$ in the classical sense. This tells us that the extended twice differentiability does not imply the classical twice differentiability.

Remark 2.3. The concept twice differentiable of function $f : \mathbb{R}^n \to \mathbb{R}$ in the extended sense stated in Definition 2.1(i) coincides with the twice differentiable of function f in the extended sense stated in [5, Definition 13.1 (b)]. However, with this concept statement in [5, Definition 13.1 (b)], because $D_f \cap D_g$ can not be the domain of $\nabla(f+g)$, $\nabla(f.g)$ and $\nabla(\frac{f}{g})$, where D_f and D_g are the domains of $\nabla(f)$ and $\nabla(g)$, respectively. This makes difficulty to construct mathematical operations for functions that are twice differentiable in the extended sense.

3 Main results

In this section, we present the results obtained on the calculation rule for the extended Hessian. Besides that, we also prove why the counterpart of the second order differentiability in the extended sense on $\mathbb{R}^n$ does not appear in variational analysis.

Proposition 3.1. *Suppose that the function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a twice differentiable at $\bar{x}$ in the extended sense. Then*

$$\nabla^2 f(\bar{x}) = \Big[\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x})\Big]_{i,j=1}^{n,n}. \tag{1}$$

Proof. Since f is twice differentiable at $\bar{x}$ in the extended sense, from Definition 2.1(ii) f has second order partial derivatives at $\bar{x}$ in the extended sense. Suppose that

$$\nabla^2 f(\bar{x}) := \Big[a_{ij}\Big]_{i,j=1}^{n,n}$$

we have

$$\lim_{x \xrightarrow{D} \bar{x}} \frac{\big[\nabla f(x)\big]^T - \big[\nabla f(\bar{x})\big]^T - \nabla^2 f(\bar{x})(x-\bar{x})}{\|x-\bar{x}\|} = 0,$$

with $D = D_1 \times \ldots \times D_n$ is a subset of U satisfies $\mu(U \setminus D) = 0$, for some neighborhood $U := U_1 \times \ldots \times U_n$ of $\bar{x}$. Therefore, $\mu(U_i \setminus D_i) = 0$ for all $i = 1, \ldots, n$ and we get

$$\begin{aligned} 0 &= \lim_{x \xrightarrow{D} \bar{x}} \frac{\Big(\frac{\partial f}{\partial x_1}(x),\ldots,\frac{\partial f}{\partial x_n}(x)\Big) - \Big(\frac{\partial f}{\partial x_1}(\bar{x}),\ldots,\frac{\partial f}{\partial x_n}(\bar{x})\Big) - \Big[a_{ij}\Big]_{i,j=1}^{n,n}.(x_1-\bar{x}_1,\ldots,x_n-\bar{x}_n)^T}{\|x-\bar{x}\|} \\ &= \lim_{x \xrightarrow{D} \bar{x}} \frac{\Big(\frac{\partial f}{\partial x_1}(x)-\frac{\partial f}{\partial x_1}(\bar{x})-a_{11}(x_1-\bar{x}_1)-\ldots-a_{n1}(x_n-\bar{x}_n),\ldots,\frac{\partial f}{\partial x_n}(x)-\frac{\partial f}{\partial x_n}(\bar{x})-a_{1n}(x_1-\bar{x}_1)-\ldots-a_{nn}(x_n-\bar{x}_n)\Big)}{\|x-\bar{x}\|} \\ &= \lim_{x \xrightarrow{D} \bar{x}} \Big(\frac{\frac{\partial f}{\partial x_1}(x)-\frac{\partial f}{\partial x_1}(\bar{x})-a_{11}(x_1-\bar{x}_1)-\ldots-a_{n1}(x_n-\bar{x}_n)}{\|x-\bar{x}\|},\ldots,\frac{\frac{\partial f}{\partial x_n}(x)-\frac{\partial f}{\partial x_n}(\bar{x})-a_{1n}(x_1-\bar{x}_1)-\ldots-a_{nn}(x_n-\bar{x}_n)}{\|x-\bar{x}\|}\Big). \end{aligned}$$

Hence, for each $i = 1, \ldots, n$ we choose $x = (\bar{x}_1, \ldots \bar{x}_{i-1}, x_i, \bar{x}_{i+1} \ldots, \bar{x}_n)$, for each $j = 1, \ldots, n$ we have

$$\lim_{x_i \xrightarrow{D_i} \bar{x}_i} \frac{\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(\bar{x}) - a_{ji}(x_i - \bar{x}_i)}{\|x_i - \bar{x}_i\|} = 0$$

Combining this with Definition 2.1 (ii), we get

$$a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) \text{ for all } i, j = 1, \ldots, n.$$

This shows that

$$\nabla^2 f(\bar{x}) = \Big[\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x})\Big]_{i,j=1}^{n,n}.$$

The proof is complete. □

Representation (1) was mentioned by Rockafellar in his paper [4] for convex functions without proof.

Theorem 3.2. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Assume that f, g are twice differentiable at $\bar{x}$ in the extended sense. Then*

(**i**) *$f + g$ is twice differentiable at $\bar{x}$ in the extended sense, with the extended Hessian matrix given by $\nabla^2(f+g)(\bar{x}) := \nabla^2(f)(\bar{x}) + \nabla^2(g)(\bar{x})$.*

(**ii**) *αf is twice differentiable at $\bar{x}$ in the extended sense, where $\alpha \in \mathbb{R}$ is a given real constant, with the extended Hessian matrix given by $\nabla^2(\alpha f)(\bar{x}) := \alpha \nabla^2(f)(\bar{x})$.*

(**iii**) *$f.g$ is twice differentiable at $\bar{x}$ in the extended sense, with the extended Hessian matrix given by $\nabla^2(f.g)(\bar{x}) := g(\bar{x})\nabla^2 f(\bar{x}) + \big[\nabla f(\bar{x})\big]^T.\nabla g(\bar{x}) + \big[\nabla g(\bar{x})\big]^T.\nabla f(\bar{x}) + f(\bar{x})\nabla^2 g(\bar{x})$.*

(**iv**) *If in addition that $g(\bar{x}) \neq 0$ then $\frac{f}{g}$ is also twice differentiable at $\bar{x}$ in the extended sense, with the Hessian matrix given by*

$$\nabla^2(\frac{f}{g}) := \frac{\nabla^2 f(\bar{x})}{g(\bar{x})} - \frac{\big[\nabla f(\bar{x})\big]^T.\nabla g(\bar{x})}{[g(\bar{x})]^2} - \frac{\big[\nabla g(\bar{x})\big]^T.\nabla f(\bar{x})}{[g(\bar{x})]^2} - f(\bar{x}).\frac{g(\bar{x})\nabla^2 g(\bar{x}) - 2\big[\nabla g(\bar{x})\big]^T \nabla g(\bar{x})}{[g(\bar{x})]^3}.$$

Proof. Since f, g are twice differentiable at $\bar{x}$ in the extended sense. We have $\mu(U \setminus D_f) = 0$, $\mu(U \setminus D_g) = 0$ for some neighborhood U of $\bar{x}$, where μ is the Lebesgue measure on $\mathbb{R}^n$. Hence,

$$0 \leq \mu\big[U \setminus \big(D_f \cap D_g\big)\big] = \mu\big[\big(U \setminus D_f \cup \big(U \setminus D_g\big)\big)\big] \leq \mu\big(U \setminus D_f\big) + \mu\big(U \setminus D_g\big) = 0,$$

which implies $\mu\big[U \setminus \big(D_f \cap D_g\big)\big] = 0$. Since f, g are locally Lipschitz at $\bar{x}$, there exist $\epsilon > 0$, $\kappa_1 > 0, \kappa_2 > 0$ such that

$$|f(x_1) - f(x_2)| \leq \kappa_1 |x_1 - x_2|,$$
$$|g(x_1) - g(x_2)| \leq \kappa_2 |x_1 - x_2|,$$

for all $x_1, x_2 \in \mathbb{B}_\epsilon(\bar{x})$. Put

$$m_1 = \min\{|f(x)| \mid x \in \mathbb{B}_\epsilon(\bar{x})\},\ M_1 = \max\{|f(x)| \mid x \in \mathbb{B}_\epsilon(\bar{x})\},$$

$$m_2 = \min\{|g(x)| \mid x \in \mathbb{B}_\epsilon(\bar{x})\},\ M_2 = \max\{|g(x)| \mid x \in \mathbb{B}_\epsilon(\bar{x})\}.$$

(**i**) $f + g$ is twice differentiable at $\bar{x}$ in the extended sense:
Indeed, since f, g are differentiable at $\bar{x}$ then $f + g$ is also differentiable at $\bar{x}$. For any $x_1, x_2 \in \mathbb{B}_\epsilon(\bar{x})$, we have

$$\big|f(x_1) + g(x_1) - \big(f(x_2) + g(x_2)\big)\big| \leq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| \leq (\kappa_1 + \kappa_2)|x_1 - x_2|.$$

This infers that $f+g$ is Lipschitz continuous on $\mathbb{B}_\epsilon(\bar{x})$. On the other hand, we have

$$\begin{aligned} 0 &\le \frac{\left\| \left[\nabla f(x)\right]^T + \left[\nabla g(x)\right]^T - \left(\left[\nabla f(\bar{x})\right]^T + \left[\nabla g(\bar{x})\right]^T\right) - \left(\nabla^2 f(\bar{x}) + \nabla^2 g(\bar{x})\right)(x-\bar{x}) \right\|}{\|x-\bar{x}\|} \\ &\le \frac{\left\| \left[\nabla f(x)\right]^T - \left[\nabla f(\bar{x})\right]^T - \nabla^2 f(\bar{x})(x-\bar{x}) \right\|}{\|x-\bar{x}\|} + \frac{\left\| \left[\nabla g(x)\right]^T - \left[\nabla g(\bar{x})\right]^T - \nabla^2 g(\bar{x})(x-\bar{x}) \right\|}{\|x-\bar{x}\|} \\ &\to 0 \text{ as } x \xrightarrow{D_f \cap D_g} \bar{x}. \end{aligned}$$

This implies that

$$\lim_{x \xrightarrow{D_f \cap D_g} \bar{x}} \frac{\left(\left[\nabla(f+g)(x)\right]^T\right) - \left(\left[\nabla(f+g)(\bar{x})\right]^T\right) - \left(\nabla^2 f(\bar{x}) + \nabla^2 g(\bar{x})\right)(x-\bar{x})}{\|x-\bar{x}\|} = 0.$$

In other words $\nabla f + \nabla g$ is differentiable at $\bar{x}$ relative to $D_f \cap D_g$. So, we show that $f+g$ is differentiable at $\bar{x}$, Lipschitz on $\mathbb{B}_\epsilon(\bar{x})$ and $\nabla(f+g)$ is differentiable at $\bar{x}$ relative to $D_f \cap D_g$. It shows that $f+g$ is twice differentiable at $\bar{x}$ in the extended sense, with its extended Hessian matrix is $\nabla^2 f(\bar{x}) + \nabla^2 g(\bar{x})$.

(**ii**) αf is twice differentiable at $\bar{x}$ in the extended sense:
Indeed, since f is differentiable at $\bar{x}$ then αf is also differentiable at $\bar{x}$. For any $x_1, x_2 \in \mathbb{B}_\epsilon(\bar{x})$, we have

$$|\alpha f(x_1) - \alpha f(x_2)| \le |\alpha| \, |f(x_1) - f(x_2)| \le |\alpha| \, \kappa_1 \, |x_1 - x_2| \, .$$

This infers that αf is Lipschitz continuous on $U = \mathbb{B}_\epsilon(\bar{x})$. On the other hand, we have

$$\lim_{x \xrightarrow{D_f} \bar{x}} \frac{\alpha\left[\nabla f(x)\right]^T - \alpha\left[\nabla f(\bar{x})\right]^T - \alpha\nabla^2 f(\bar{x})(x-\bar{x})}{\|x-\bar{x}\|} = \alpha \lim_{x \xrightarrow{D_f} \bar{x}} \frac{\left[\nabla f(x)\right]^T - \left[\nabla f(\bar{x})\right]^T - \nabla^2 f(\bar{x})(x-\bar{x})}{\|x-\bar{x}\|} = 0.$$

Therefore, αf is twice differentiable at $\bar{x}$ in the extended sense, with its extended Hessian matrix is $\alpha\nabla^2 f(\bar{x})$.

(**iii**) $f.g$ is twice differentiable at $\bar{x}$ in the extended sense:
Indeed, since f, g are differentiable at $\bar{x}$, $f.g$ is also differentiable at $\bar{x}$. For any $x_1, x_2 \in \mathbb{B}_\epsilon(\bar{x})$, we have

$$\begin{aligned} |f(x_1).g(x_1) - f(x_2).g(x_2)| &\le |f(x_1).g(x_1) - f(x_1).g(x_2)| + |f(x_1).g(x_2) - f(x_2).g(x_2)| \\ &= |f(x_1)| \, |g(x_1) - g(x_2)| + |g(x_2)| \, |f(x_1) - f(x_2)| \\ &\le (M_1.\kappa_2 + M_2\kappa_1) \, |x_1 - x_2| \, . \end{aligned}$$

This shows that $f.g$ is Lipschitz on $U = \mathbb{B}_\epsilon(\bar{x})$. Moreover, for

$$A := g(\bar{x})\nabla^2 f(\bar{x}) + \left[\nabla f(\bar{x})\right]^T.\nabla g(\bar{x}) + \left[\nabla g(\bar{x})\right]^T.\nabla f(\bar{x}) + f(\bar{x})\nabla^2 g(\bar{x})$$

we have

$$\begin{aligned} &\frac{f(x).\left[\nabla g(x)\right]^T + g(x)\left[\nabla f(x)\right]^T - f(\bar{x}).\left[\nabla g(\bar{x})\right]^T - g(\bar{x})\left[\nabla f(\bar{x})\right]^T - A(x-\bar{x})}{\|x-\bar{x}\|} \\ &= \frac{f(x)\left[\nabla g(x)\right]^T - f(\bar{x})\left[\nabla g(\bar{x})\right]^T - \left[\left[\nabla g(\bar{x})\right]^T.\nabla f(\bar{x}) + f(\bar{x})\nabla^2 g(\bar{x})\right](x-\bar{x})}{\|x-\bar{x}\|} \\ &\quad + \frac{g(x)\left[\nabla f(x)\right]^T - g(\bar{x})\left[\nabla f(\bar{x})\right]^T - \left[\left[\nabla f(\bar{x})\right]^T.\nabla g(\bar{x}) + g(\bar{x})\nabla^2 f(\bar{x})\right](x-\bar{x})}{\|x-\bar{x}\|}. \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned}
&\frac{f(x)\big[\nabla g(x)\big]^T - f(\bar{x})\big[\nabla g(\bar{x})\big]^T - \Big[\big[\nabla g(\bar{x})\big]^T.\nabla f(\bar{x}) + f(\bar{x})\nabla^2 g(\bar{x})\Big](x-\bar{x})}{\|x-\bar{x}\|}\\
&= \frac{\big[\nabla g(x)\big]^T\big[f(x)-f(\bar{x})-\nabla f(\bar{x})(x-\bar{x})\big]}{\|x-\bar{x}\|} + \frac{\Big(\big[\nabla g(x)\big]^T - \big[\nabla g(\bar{x})\big]^T\Big)\nabla f(\bar{x})(x-\bar{x})}{\|x-\bar{x}\|}\\
&\quad + \frac{f(\bar{x})\Big[\big[\nabla g(x)\big]^T - \big[\nabla g(\bar{x})\big]^T - \nabla^2 g(\bar{x})(x-\bar{x})\Big]}{\|x-\bar{x}\|}\\
&\to 0 \text{ as } x \overset{D_f\cap D_g}{\longrightarrow} \bar{x}.
\end{aligned} \tag{3}$$

Similarly, we also have

$$\begin{aligned}
&\frac{g(x)\big[\nabla f(x)\big]^T - g(\bar{x})\big[\nabla f(\bar{x})\big]^T - \Big[\big[\nabla f(\bar{x})\big]^T.\nabla g(\bar{x}) + g(\bar{x})\nabla^2 f(\bar{x})\Big](x-\bar{x})}{\|x-\bar{x}\|}\\
&\to 0 \text{ as } x \overset{D_f\cap D_g}{\longrightarrow} \bar{x}.
\end{aligned} \tag{4}$$

From (2), (3) and (4) it follows

$$\lim_{x \overset{D_f\cap D_g}{\longrightarrow} \bar{x}} \frac{f(x).\big[\nabla g(x)\big]^T + g(x)\big[\nabla f(x)\big]^T - f(\bar{x}).\big[\nabla g(\bar{x})\big]^T - g(\bar{x})\big[\nabla f(\bar{x})\big]^T - A(x-\bar{x})}{\|x-\bar{x}\|} = 0.$$

Therefore, $f.g$ is twice differentiable at $\bar{x}$ in the extended sense, with the extended Hessian matrix is $A = g(\bar{x})\nabla^2 f(\bar{x}) + \big[\nabla f(\bar{x})\big]^T.\nabla g(\bar{x}) + \big[\nabla g(\bar{x})\big]^T.\nabla f(\bar{x}) + f(\bar{x})\nabla^2 g(\bar{x})$.

(**iv**) $\frac{f}{g}$ is twice differentiable at $\bar{x}$ in the extended sense. Firstly, we prove that $\frac{1}{g}$ is twice differentiable at $\bar{x}$ in the extended sense, with the extended Hessian matrix is $\frac{g(\bar{x})\nabla^2 g(\bar{x}) - 2[\nabla g(\bar{x})]^T\nabla g(\bar{x})}{[g(\bar{x})]^3}$. Indeed, since g is differentiable at $\bar{x}$ and $g(\bar{x}) \neq 0$, we have known that $\frac{1}{g}$ is also differentiable at $\bar{x}$. Without loss of generality, assume $g(x) \neq 0$ for all $x \in \mathbb{B}_\epsilon(\bar{x})$. Then we have $m_2 > 0$. For any $x_1, x_2 \in \mathbb{B}_\epsilon(\bar{x})$, we have

$$\left|\frac{1}{g(x_1)} - \frac{1}{g(x_2)}\right| = \frac{|g(x_1) - g(x_2)|}{|g(x_1)|\,.\,|g(x_2)|} \leq \frac{\kappa_2}{m_2^2}\,|x_1 - x_2|\,.$$

This shows that $\frac{1}{g}$ is Lipschitz on $\mathbb{B}_\epsilon(\bar{x})$. On the other hand, for $B := \frac{g(\bar{x})\nabla^2 g(\bar{x}) - 2[\nabla g(\bar{x})]^T\nabla g(\bar{x})}{[g(\bar{x})]^3}$ we have

$$\begin{aligned}
&\frac{\frac{\big[\nabla g(x)\big]^T}{[g(x)]^2} - \frac{\big[\nabla g(\bar{x})\big]^T}{[g(\bar{x})]^2} - B(x-\bar{x})}{\|x-\bar{x}\|}\\
&= \frac{[g(\bar{x})]^3\big[\nabla g(x)\big]^T - g(\bar{x})[g(x)]^2\big[\nabla g(\bar{x})\big]^T - g(\bar{x})[g(x)]^2\nabla^2 g(\bar{x})(x-\bar{x}) + 2[g(x)]^2[\nabla g(\bar{x})]^T\nabla g(\bar{x})(x-\bar{x})}{[g(x)]^2[g(\bar{x})]^3\|x-\bar{x}\|}\\
&= \frac{1}{[g(\bar{x})]^2}.\frac{\big[\nabla g(x)\big]^T - \big[\nabla g(\bar{x})\big]^T - \nabla^2 g(\bar{x})(x-\bar{x})}{\|x-\bar{x}\|} - \frac{2\big[\nabla g(\bar{x})\big]^T}{[g(\bar{x})]^3}.\frac{g(x)-g(\bar{x})-\nabla g(\bar{x})(x-\bar{x})}{\|x-\bar{x}\|}\\
&+ \frac{1}{[g(x)]^2[g(\bar{x})]^3}.\frac{g(\bar{x})-g(x)}{\|x-\bar{x}\|}.\Big[[g(\bar{x})]^2\big[\nabla g(x)\big]^T + g(\bar{x})g(x)\big[\nabla g(x)\big]^T - 2[g(x)]^2\big[\nabla g(\bar{x})\big]^T\Big]\\
&\to 0 \text{ as } x \overset{D_g}{\to} \bar{x}.
\end{aligned}$$

Which implies that

$$\lim_{x \xrightarrow{D_g} \bar{x}} \frac{\frac{\left[\nabla g(x)\right]^T}{[g(x)]^2} - \frac{\left[\nabla g(\bar{x})\right]^T}{[g(\bar{x})]^2} - B(x-\bar{x})}{\|x-\bar{x}\|} = 0.$$

Thus $-\frac{\nabla g}{[g(\bar{x})]^2}$ is differentiable at $\bar{x}$ relative to D_g. Therefore, $\frac{1}{g}$ is twice differentiable at $\bar{x}$ in the extended sense and its extended Hessian is $-B$.

Finally, since $f, \frac{1}{g}$ are twice differentiable at $\bar{x}$ in the extended sense, by (**iii**) we get $\frac{f}{g}$ is twice differentiable at $\bar{x}$ in the extended sense, with the Hessian matrix is $\nabla^2(\frac{f}{g}) := \frac{\nabla^2 f(\bar{x})}{g(\bar{x})} - \frac{\left[\nabla f(\bar{x})\right]^T.\nabla g(\bar{x})}{[g(\bar{x})]^2} - \frac{\left[\nabla g(\bar{x})\right]^T.\nabla f(\bar{x})}{[g(\bar{x})]^2} - f(\bar{x}).\frac{g(\bar{x})\nabla^2 g(\bar{x}) - 2\left[\nabla g(\bar{x})\right]^T\nabla g(\bar{x})}{[g(\bar{x})]^3}$. The proof is complete. □

Lemma 3.3. ([1, Theorem 2.3.7]). *Let f be a Lipschitz function on an open subset of $\mathbb{R}^n$ containing the line segment $[x,y]$ with $x, y \in \mathbb{R}^n$, $x \neq y$. Then, there exists $c \in (x,y)$ such that*

$$f(y) - f(x) \in \left\langle \partial^{Cl} f(c), y - x \right\rangle. \tag{5}$$

Lemma 3.4. ([1, Theorem 2.5.1]). *Let f be Lipschitz on a neighborhood U of $\bar{x} \in \mathbb{R}^n$, and differentiable on a subset D of U with $\mu(U \backslash D) = 0$, where μ is the Lebesgue measure on $\mathbb{R}^n$. Then, one has*

$$\partial^{Cl} f(\bar{x}) = \mathrm{co}\Big\{\lim_{n\to\infty} \nabla f(x_k) |\ x_k \xrightarrow{D} \bar{x} \text{ and } \lim_{k\to\infty} \nabla f(x_k) \text{ exists}\Big\}.$$

From definitions of the second order differentiability at a point in the extended sense, it is natural to define the second order differentiability at every point in the extended sense as follows: A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *twice differentiable on $\mathbb{R}^n$ in the extended sense* if it is twice differentiable at every point $x \in \mathbb{R}^n$ in the extended sense.

In contrast to the extended second order differentiability, it turns out that the extended second order differentiability coincides with the classical one.

Theorem 3.5. *If a function $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable at every point $x \in \mathbb{R}^n$ in the extended sense then f is twice differentiable at every point $x \in \mathbb{R}^n$ in the classical sense.*

Proof. Indeed, since f is twice differentiable at every point $x \in \mathbb{R}^n$ in the extended sense, then f is differentiable at every point $x \in \mathbb{R}^n$. It means that f is differentiable on $\mathbb{R}^n$. Moreover, there exist a matrix $n \times n$ A, a neighborhood U of x and subset D of U with $\mu(U \setminus D) = 0$ such that

$$\lim_{u \xrightarrow{D} x} \frac{\left[\nabla f(u)\right]^T - \left[\nabla f(x)\right]^T - A(u-x)}{\|u-x\|} = 0 \tag{6}$$

where μ denotes the Lebesgue measure on $\mathbb{R}^n$. Assume without loss of generality that U is an open convex set. Since f is Lipschitz on U, by Lemma 3.4, we have

$$\partial^{Cl} f(x) = \mathrm{co}\Big\{\lim_{k\to\infty} \nabla f(x_k) |\ x_k \xrightarrow{D} x \text{ such that } \lim_{k\to\infty} \nabla f(x_k) \text{ exists}\Big\}, \tag{7}$$

We then show that

$$\lim_{u\to x}\frac{\big[\nabla f(u)\big]^T-\big[\nabla f(x)\big]^T-A(u-x)}{\|u-x\|}=0.$$

Take any $\varepsilon>0$. By (6) there exists an open convex neighborhood V of $\bar{x}$ such that $V\subset U$ and

$$\|\big[\nabla f(u')\big]^T-\big[\nabla f(x)\big]^T-A(u'-x)\|\le\varepsilon\|u'-x\|\quad\text{for all } u'\in D\cap V. \tag{8}$$

Let $u\in V$. Noting that $V\subset U$ and $\nabla f(x)\in\partial^{Cl}f(x)$, $\nabla f(u)\in\partial^{Cl}f(u)$, by (7), there exist $s\in\mathbb{N}^*$, $\alpha^i,\beta^j\in[0,1]$, $x_k^i\in D\cap V$, $u_k^j\in D\cap V$ for every $i,j=1,\dots,s$ and $k\in\mathbb{N}^*$ such that

$$\sum_{i=1}^s\alpha^i=1,\ \sum_{j=1}^s\beta^j=1,\lim_{k\to\infty}x_k^i=x,\lim_{k\to\infty}u_k^j=u,$$

and $\nabla f(x)=\sum\limits_{i=1}^s\alpha^iv_x^i,\ \nabla f(u)=\sum\limits_{j=1}^s\beta^jv_u^j$ with $v_x^i=\lim\limits_{k\to\infty}\nabla f(x_k^i), v_u^j=\lim\limits_{k\to\infty}\nabla f(u_k^j)$. This together with (8) implies that

$$\begin{aligned}
0\ &\le\|\big[\nabla f(x)\big]^T-\big[\nabla f(u)\big]^T-A(x-u)\|\\
&=\Big\|\sum_{i=1}^s\alpha^iv_x^i-\sum_{j=1}^s\beta^jv_u^j-A(x-u)\Big\|\\
&=\Big\|\sum_{i=1}^s\sum_{j=1}^s\alpha^i\beta^j\big(v_x^i-v_u^j-A(x-u)\big)\Big\|\\
&\le\max_{i,j\in\{1,\dots,s\}}\big\|v_x^i-v_u^j-A(x-u)\big\|\\
&=\max_{i,j\in\{1,\dots,s\}}\Big\|\lim_{k\to\infty}\big(\nabla f(x_k^i)-\nabla f(u_k^j)-A(x_k^i-u_k^j)\big)\Big\|\\
&\le\max_{i,j\in\{1,\dots,s\}}\lim_{k\to\infty}\big(\varepsilon\|x_k^i-u_k^j\|\big)\\
&=\varepsilon\|x-u\|.
\end{aligned}$$

Therefore, ∇f is differentiable at x. This means that f is twice differentiable at x. Because $x\in\mathbb{R}^n$ is taken any, f is twice differentiable on $\mathbb{R}^n$. □

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TÓM TẮT

MỘT SỐ KẾT QUẢ VỀ SỰ KHẢ VI BẬC HAI THEO NGHĨA MỞ RỘNG CỦA HÀM SỐ

Hà Anh Tuấn
Khoa Khoa học cơ bản, Trường Đại học Giao thông vận tải TP Hồ Chí Minh
Ngày nhận bài 30/6/2021, ngày nhận đăng 28/9/2021

Trong bài báo này, chúng tôi giới thiệu khái niệm đạo hàm riêng bậc hai theo nghĩa mở rộng cho các hàm không lồi và chứng minh một công thức tính toán các phần tử của Hessian mở rộng theo các đạo hàm riêng cấp hai mở rộng. Chúng tôi cho thấy rằng tổng, hiệu, tích và thương của các hàm khả vi hai lần theo nghĩa mở rộng tại cùng một điểm là các hàm khả vi hai lần tại điểm đó theo nghĩa mở rộng. Chúng tôi cũng cho thấy vì sao khái niệm khả vi hai lần theo nghĩa mở rộng trên $\mathbb{R}^n$ không xuất hiện trong phân tích biến phân.

Từ khóa: Khả vi hai lần theo nghĩa mở rộng; đạo hàm riêng bậc hai theo nghĩa mở rộng.